Discrete Applied Mathematics 13 (1986) 101–104
North-Holland

COMMUNICATION

TESTING HOMOTOPY EQUIVALENCE IS ISOMORPHISM COMPLETE

Yechezkel ZALCSTEIN and Stanley P. FRANKLIN
Department of Mathematical Sciences, Memphis State University, Memphis, TN 38152, USA

Communicated by R.M. Karp
Received 27 August 1985

It is shown that homotopy equivalence of finite topological spaces is polynomially equivalent to testing graph isomorphism.

1. Introduction

A problem that is polynomial time equivalent to graph isomorphism is known as *isomorphism complete* ([1], [2], [3], [5], [6]).

Most of the known isomorphism complete problems (see [2]) are isomorphism problems for restricted classes of graphs. Booth [1] has attempted to find isomorphism complete problems that are 'substantially' different. He found some problems for semigroups and automata, but these were still isomorphism problems. Colbourn [3] has discovered isomorphism complete problems concerning matrices that are not 'isomorphism' problems. Other isomorphism complete problems that are not 'isomorphism' problems are due to Mathon [6]. These include finding a set of generators for, and the cardinality of, the authomorphism group of a graph.

In [5], a somewhat different kind of isomorphism complete problem was found - homeomorphism of finite topological spaces. However, this was accomplished by exhibiting an equivalence, computable in polynomial time, between the category of finite topological spaces and the category of finite transitive digraphs.

Herein we exhibit an isomorphism complete problem that is very different. It involves testing if two finite topological spaces are homotopy equivalent. A homotopy is, informally, a continuous deformation of topological spaces. If one regards a space as a digraph, as indicated in the previous paragraph, homotopy does not preserve, in general, any of the classical graph theoretic invariants and hence homotopy equivalence is much weaker than homeomorphism. Thus testing homotopy equivalence is a problem that is genuinely different from testing isomorphism.

Homotopy equivalence has useful ties with more combinational properties of posets which will be explored in a forthcoming paper.

2. Topological preliminaries

Recall that a topological space is a set, X, together with a collection, $T(X)$, of subsets of X, called the *topology* of X. The members of $T(X)$ are called the *open* sets of X. The empty set and X itself are always open sets. $T(X)$ is closed under finite intersections and arbitrary unions. A topological space is T_0 if for any two distinct points there is an open set containing one but not the other.

A function, or map, between topological spaces is called *continuous* if the inverse image of each open set is open. A *homeomorphism* of topological spaces is a continuous bijection with a continuous inverse.

Let I denote the closed unit interval $[0,1]$. Let $f, g: X \to Y$ be continuous functions. A *homotopy* between f and g is a continuous function $H: X \times I \to Y$ such that $H(x,0) = f(x)$ and $X(x,1) = g(x)$. Two continuous functions $f, g: X \to Y$ are called *homotopic* (denoted $f \sim g$) if there is a homotopy between them. $\sim$ is easily seen to be an equivalence relation. Intuitively, a homotopy is a continuous 'deformation' of f into g.

Two topological spaces X and Y are called *homotopy equivalent* if there are functions $f: X \to Y$ and $g: Y \to X$ such that $gf \sim I_X$ and $fg \sim I_y$ where I denotes the identity function on the respective space.

An example of a pair of homotopically equivalent spaces is the punctured disk (the open unit ball in the plane with a point removed) and the unit circle.

For all undefined topological concepts see [4].

3. Finite topological space

In this section we review the relevant results from [5] and [7].

Let X be a finite topological space and let x be a point in X. Since there are only finitely many open sets containing x, their intersection $M(X)$ is open (called the *minimal open* set of x). If X is a T_0 space, then for any two distinct points either $y \notin M(x)$ or $y \notin M(y)$. Thus the relation $x < y$ iff $x \in M(y)$ is asymmetric, and is easily seen to be a partial order. Conversely, given a poset $P = (X, <)$, $M(x) = \{y \mid y < x\}$ defines a collection of minimal open sets for a topology on X. Furthermore, the transformation between posets and T_0 spaces can be done in polynomial time. Thus we will speak interchangeably of posets and T_0 spaces.

Definition. Let $(X, <)$ be a poset. $x \in X$ is called *linear* if there exists $y > x$ such that for all $z > x$, $z \geq y$. Dually, x is *colinear* if there exists $y < x$ such that for all $z < x$, $z \leq y$. A finite poset is a *core* if it has no linear or colinear points.

A *core of a poset* $(X, <)$ is a subposet X' of X such that X' is a core and such that X' (as a subspace of the T_0 space X) is homotopically equivalent to X.

Theorem 1 [7]. *Every finite poset has a core.*

Furthermore if X, Y are finite posets with cores X', Y', then X is homotopy equivalent to Y if and only if X' is homeomorphic to Y'.

The core $C(X)$ can be computed by the following simple algorithm

Algorithm C
$C(X) := X$;
while X has a linear or colinear point x **do**
$C(X) := X - \{x\}$;

4. Complexity of testing homotopy equivalence

Theorem 2. *Testing homotopy equivalence is isomorphism complete.*

Proof. Let X, Y be finite T_0 spaces. By Theorem 1, X and Y are homotopy equivalent if and only if their cores are homeomorphic. (Note that Algorithm C may give different values for $C(X)$ depending on the order on which the points are removed. However, any two values are homeomorphic, by Theorem 1.) Algorithm C runs in polynomial time. Thus testing homotopy equivalence is polynomially equivalent to testing homeomorphism of cores or equivalently, to testing isomorphism of posets without linear or colinear points.

Let $(X, <)$ be an arbitrary finite poset. Consider X as a digraph with edges (x, y) iff $y<x$. For each vertex x in X, add four new vertices x^1, x^2, x_1, x_2 and edges $(x^1, x), (x^2, x), (x, x_1), (x, x_2)$. Let $\bar{X}$ be the transitive closure of the extended digraph. By construction, $\bar{X}$ has no linear or colinear points and $\bar{x}$ can be constructed from X in polynomial time.

Let ϕ be an isomorphism between $\bar{X}$ and $\bar{Y}$. Since points in $\bar{Y} - Y$ are maximal or minimal, and since points in X, (respectively Y) can not by construction be maximal or minimal in $\bar{X}$ (respectively $\bar{Y}$) ϕ must map points in X into points in Y. So ϕ restricted to X is an isomorphism of X and Y.

Thus the transformation $X \to \bar{X}$ is a polynomial time reduction of poset isomorphism to isomorphism of cores. Since poset isomorphism is isomorphism complete, the theorem follows.

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